

# A Time Dependent Multi-Determinant approach to nuclear dynamics.

G. Puddu

Dipartimento di Fisica dell'Universita' di Milano,

Via Celoria 16, I-20133 Milano, Italy

August 2, 2012

## Abstract

We study a multi-determinant approach to the time evolution of the nuclear wave functions (TDMD). We employ the Dirac variational principle and use as ansatz for the nuclear wave-function a linear combination of Slater determinants and derive the equations of motion. We demonstrate explicitly that the norm of the wave function and the energy are conserved during the time evolution. This approach is a direct generalization of the time dependent Hartree-Fock method. We apply this approach to a case study of  ${}^6\text{Li}$  using the N3LO interaction renormalized to 4 major harmonic oscillator shells. We solve the TDMD equations of motion using Krylov subspace methods of Lanczos type. We discuss as an application the isoscalar monopole strength function.

**Pacs numbers:** 21.60.-n, 24.10.Cn, 31.70.Hq

# 1 Introduction.

The time dependent Hartree-Fock method (TDHF) and its quasi-particle generalization, the time dependent Hartree-Fock-Bogoliubov method (TDHFB), are central tools in studying nuclear dynamics (see for example ref. [1]), whereby the nuclear wave function is approximated by a single Slater determinant or by a quasi-particle determinant wave-function. In the small amplitude limit they reduce to the random phase approximation (RPA) and to the quasi-particle random phase approximation (QRPA) which are commonly used to describe nuclear excitation, as for example giant resonances.

Despite the enormous matrix dimensions, these limits are nowadays solved using efficient Krylov projection techniques of Arnoldi type (see for example ref.[2] for recent applications). Recently the time dependent coupled-cluster method (refs. [3],[4]) has been revisited and applied to light nuclei using a similarity group transformed N3LO interaction (ref. [5]).

In this work we discuss a Time Dependent Multi-Determinant (TDMD) approach whereby the nuclear wave function is approximated by a linear combination of Slater determinants. This approach is the time dependent version of the Hybrid Multi-Determinant (HMD) approach (refs. [6],[7]). To the author knowledge, this approach has never been considered in nuclear physics. In this sense this is an exploratory study. Our starting point is the Dirac variational principle which, as well known, leads to the time-dependent Schrodinger equation or the time-dependent equations of motion as the TDHF (ref.[8]) within the approximation of the nuclear wave function with a single Slater determinant. Within the Dirac variational principle we derive the equations of motion and demonstrate explicitly that the time evolution conserves the norm of the wave function and the

energy. Since the equations of motion are, as we shall see, of the type  $iL\dot{\psi} = R$  where  $R$  is an energy gradient  $\psi$  is a sum of Slater determinants and  $L$  is a matrix (of rather large dimensions) embodying the time-derivative of the norm of the wave-function (which will be discussed in detail below), the actual evaluation of the wave function as a function of time is performed using the Direct Lanczos method (DL) for the solution of a rather large linear system. The DL method is a Krylov subspace type of method to solve linear systems (an excellent review of these methods can be found for example in ref. [9]). The eigenvalue version of these methods are the familiar Lanczos method used in the shell model approach to nuclear structure and the Arnoldi method used in solving the RPA or QRPA eigenvalue problems. The basic idea of these methods is that although we cannot store a matrix (e.g. the nuclear Hamiltonian matrix) we can evaluate the matrix to vector product easily. Although  $L$  is not as large as the shell model Hamiltonian matrix it can hardly be stored except in very simple cases. However the matrix to vector product appearing in the equations of motion is trivial to evaluate, and therefore the Lanczos method is the ideal one. We solve the equations of motion, as an exploratory study, in the case of  ${}^6\text{Li}$  using the N3LO interaction renormalized to 4 major oscillator shells with the Lee-Suzuki (ref.[10],[11]) method, in order to reduce the otherwise very large single-particle space. We use the time-dependent wave function thus obtained to evaluate strength functions. Our ultimate goal is to extend ab-initio methods to time-dependent problems, such as the evaluation of strength functions, starting from the nucleon-nucleon interaction.

The outline of this paper is as follows. In section 2 we derive the equations of motion using the Dirac variational principle and prove that these equations of motion conserve the norm and the energy of the nuclear wave function. In section

3 we discuss the numerical method and in section 4 we discuss the application to the nuclear strength function using the boost method.

## 2 The time dependent variational principle.

The Dirac time-dependent variational principle states that the time evolution of the nuclear wave function is obtained by varying the action

$$S_1 = \int_{t_1}^{t_2} dt \mathcal{L}_1 = \int_{t_1}^{t_2} dt [i\hbar \langle \psi | \dot{\psi} \rangle - \langle \psi | \hat{H} | \psi \rangle] \quad (1a)$$

or equivalently

$$S_2 = \int_{t_1}^{t_2} dt \mathcal{L}_2 = \int_{t_1}^{t_2} dt [-i\hbar \langle \dot{\psi} | \psi \rangle - \langle \psi | \hat{H} | \psi \rangle] \quad (1b)$$

with respect to  $|\psi\rangle$  and  $\langle\psi|$  independently, under the constraint that the wave function is held fixed at the initial and final times  $t_1$  and  $t_2$ . The most general wave function in the Hilbert space will then give the Schrodinger equation and its complex conjugate. The TDHF approximation is obtained if the wave function is assumed to be a Slater determinant. In what follows we drop  $\hbar$  with the understanding that the unit of time is  $1MeV^{-1} \simeq 6.6 \times 10^{-22} sec$ . We consider the ansatz

$$|\psi\rangle = \sum_{S=1}^{N_w} |U_S\rangle \quad (2)$$

where  $|U_S\rangle$  is a Slater determinant and  $N_w$  is the total number (space dimension). These Slater determinants are of the most generic type and are written as

$$|U_S\rangle = c_{1S}^\dagger c_{2S}^\dagger \dots c_{AS}^\dagger |0\rangle \quad (3)$$

$A$  being the number of particles,  $S$  labels the Slater determinant and

$$c_{\alpha S}^\dagger = \sum_{i=1, N_s} U_{i, \alpha S} a_i^\dagger, \quad (\alpha = 1, 2, \dots, A) \quad (4)$$

in the above equation,  $a_i^\dagger$  is the creation operator in the single-particle (harmonic oscillator) state  $i$ ,  $N_s$  is the number of the single-particle states and  $U$  is the single-particle wave-function in the h.o. representation. We assume that neutrons and protons are not mixed, i.e that each Slater determinant is a product of a neutron and a proton Slater determinant. The ansatz (3)-(4) is the same as in the Hybrid multi-determinant method (refs.[6],[7]). Here we omit the projector to good quantum numbers (angular momentum and parity) solely for clarity. In principle we could apply the variational principle including projectors to good quantum numbers. The Slater determinants are not orthogonal to each other and are 'deformed', that is they do not have good quantum numbers. At the initial time they could be the result of a partially converged variational calculation as given by the HMD method. The coefficients of the linear combination in eq.(3) are set to 1 since they can be included in the definition of the  $U_{i, \alpha S}$ .

We seek to find the equations of motion for the single-particle wave functions  $U$ . In what follows, the complex conjugate of the Slater determinant  $|U_S\rangle$  is denoted as  $\langle 0|c_{AS'} \dots c_{1S'}$ , where

$$c_{\alpha, S'} = \sum_i V_{\alpha, i S'} a_i \quad (5)$$

and  $V$  is the hermitian conjugate of the matrix  $U$ . We do this in order to use simple matrix notations. Hence the indices of  $U$  are  $i, \alpha, S$  and the indices of  $V$  are  $\alpha, i, S'$ . In the following greek letters will label particles and latin letters single-particle states. The symbols  $S$  and  $S'$  will label different Slater determinants. The

Dirac variational principle gives (the bra will be denoted as  $\langle V|$ )

$$i \sum_S \delta_{V(S')} \langle V_{S'} | \dot{U}_S \rangle = \delta_V(S') \sum_S \langle V'_S | \hat{H} | U_S \rangle \quad (5a)$$

$$-i \sum_{S'} \delta_{U(S)} \langle \dot{V}_{S'} | U_S \rangle = \delta_{U(S)} \sum_{S'} \langle V'_S | \hat{H} | U_S \rangle \quad (5b)$$

where we we shown explicitly the quantities which are varied. In what follows we quote the results for the overlaps and for the matrix elements of the Hamiltonian (cf. ref.[6]). The Hamiltonian is

$$\hat{H} = \frac{1}{2} \sum_{ijkl} H_{ijkl} a_i^\dagger a_j^\dagger a_l a_k \quad (6)$$

where we recast the one-body term into the two-body interaction, as done in shell model calculations and the matrix elements  $H$  are antisymmetrized (i.e.  $H_{ijkl} = -H_{ijlk}$ ). For any  $V$  and  $U$ , (relative to the Slater determinants  $S'$  and  $S$ ) let us define

$$G = (VU)^{-1}, \quad W = GV, \quad X = UG, \quad \rho = UGV, \quad F = 1 - \rho \quad (7)$$

The matrix  $G$  has indices  $\alpha, \beta = 1, 2, \dots, A$ . The matrix  $W$  has indices  $\alpha, i$ , the matrix  $X$  has indices  $i, \alpha$  while  $\rho$  and  $F$  have indices  $i, j = 1, 2, \dots, N_s$ . The matrix  $\rho$  is the generalization of the density matrix in TDHF and satisfies the relations  $\text{tr} \rho = A$  and  $\rho^2 = \rho$  for any  $S'$  and  $S$ , as it can easily be verified. Then (cf. (ref.[6])) we have

$$\langle V | U \rangle = \det(VU) \quad (8)$$

$$\langle V | \hat{H} | U \rangle = \langle V | U \rangle \text{tr}(\Gamma \rho) \quad (9)$$

where the matrix  $\Gamma$  is given by

$$\Gamma_{ij} = \sum_{pq} H_{piqj} \rho_{qp} \quad (10)$$

Let us note that the exchange term is the same of the direct since the matrix elements are antisymmetrized. Let us note that the arrays in eqs.(7)-(10) are evaluated with  $V_{S'}$  and  $U_S$ . Also let us note that the single particle wave functions are not orthogonal to each other. Only in the case of a single Slater determinant they can be made orthogonal. The equations of motion eqs.(5a),(5b) (EOM1 and EOM2) can now be derived using the matrix identity

$$\delta \det(M) = \det(M) \text{tr}(M^{-1} \delta M) \quad (11)$$

Then it is easy to verify that

$$\langle V | \dot{U} \rangle = \langle V | U \rangle \text{tr}(G V \dot{U}) \quad (12a)$$

$$\langle \dot{V} | U \rangle = \langle V | U \rangle \text{tr}(G \dot{V} U) \quad (12b)$$

and that the explicit form for EOM1 is (using the identity  $\delta M^{-1} = -M^{-1} \delta M M^{-1}$ )

$$i \sum_{r\mu, S} \det(VU) (X_{i\alpha} W_{\mu r} + F_{ir} G_{\mu\alpha}) \dot{U}_{r\mu S} = \sum_S \det(VU) (X_{i\alpha} \mathcal{E} + 2(F\Gamma X)_{i\alpha}) \quad (13)$$

where  $\mathcal{E}$  is the energy functional

$$\mathcal{E} = \text{tr}(\Gamma \rho) \quad (14)$$

The equation of motion EOM2 can be obtained in the same way and is given by

$$-i \sum_{\mu, r, S'} \det(VU) (W_{\alpha i} X_{r\mu} + G_{\alpha\mu} F_{ri}) \dot{V}_{\mu r S'} = \sum_S \det(VU) (\mathcal{E} W_{\alpha i} + 2(W\Gamma F)_{\alpha i}) \quad (15)$$

These equations need a few comments. First, if we recast then in a schematic matrix notation

$$iL^{(1)} \dot{U} = R^{(1)} \quad (16a)$$

$$-iL^{(2)} \dot{V} = R^{(2)} \quad (16b)$$

the dimension of the linear systems to be solved can be rather large. For example In the case of  $^{24}\text{Mg}$  with 7 major shells ( $N_s = 168$ ) for 10 Slater determinants, the matrix  $L$  is  $20160 \times 20160$  (for both neutrons and protons), for many more major shells or heavier nuclei, the storage of this array in the computer RAM is a problem. Moreover these matrices seem to have a separable structure.  $L^{(1)}$  for example contains a separable term in the  $(i\alpha)(\mu r)$  indices and another term separable in the  $(ir)(\mu\alpha)$  indices. This implies that although we may not be able to store the matrix  $L$ , we can very easily perform the matrix to vector product. We only needs to store the matrices  $X, W, F$  and  $G$ , in the above  $\text{Mg}$  case of dimension  $168 \times 12, 12 \times 168$  and  $168 \times 168$  for every Slater determinant. In the past few decades, linear systems of this kind, where the matrix cannot be stored but we can easily compute matrix to vector product have received a major attention in applied mathematics using the so called Krylov subspace techniques. It is precisely the same kind of techniques one uses in standard shell model calculations. They will be briefly recalled in the next section. A systematic treatment can be found in ref. [9] (note however that in ref [9] the convention for the scalar product is  $\langle x|y \rangle = \sum x_i y_i^*$ ). Equations of motions EOM1 and EOM2 are equivalent. The matrix  $L^{(1)}$  and  $L^{(2)}$  are hermitian.

One can show that the norm of the wave function is preserved during time evolution. From eqs. (8) and (11) one has

$$d \langle \psi | \psi \rangle / dt = \sum_{SS'} \langle V | U \rangle \text{tr}[G(\dot{V}U + V\dot{U})] \quad (17)$$

with the understanding that  $S'$  refers to  $V$  and  $S$  to  $U$ . From EOM1 eq. (13), multiplying by  $V_{\alpha i S'}$  and summing over the indices one has

$$i \sum_{S'S} \langle V | U \rangle [\text{tr}(\rho) \text{tr}(GV\dot{U}) + \text{tr}(F\dot{U}GV)] =$$



$$\sum_{S'S} \langle V|U \rangle [\text{tr}(\rho)\mathcal{E} + 2\text{tr}(F\Gamma\rho)] \quad (18)$$

From EOM2 of eq. (15), multiplying by  $U_{i\alpha S}$  and summing over the indices one has

$$-i \sum_{SS'} \langle V|U \rangle [\text{tr}(\rho)\text{tr}(G\dot{V}U) + \text{tr}(X\dot{V}F)] = \sum_{S'S} \langle V|U \rangle [\text{tr}(\rho)\mathcal{E} + 2\text{tr}(\rho\Gamma F)] \quad (19)$$

Subtracting eqs. (17) and (19), since for any  $SS'$   $\text{tr}(\rho) = A$ ,

$$i \sum_{SS'} \langle V|U \rangle [A \text{tr}[G(\dot{V}U + V\dot{U})] + \text{tr}(F\dot{U}W + X\dot{V}F)] = \sum_{S'S} \langle V|U \rangle 2\text{tr}(F\Gamma\rho - \rho\Gamma F) \quad (20)$$

The right hand side of this equation is 0 since  $F = 1 - \rho$ . Next, since  $\text{tr}(\dot{\rho}) = 0$  for any  $S'S$  using the definition of  $\rho$  given in eq.(7) and the cyclic property of the trace, we have

$$\text{tr}[G(\dot{V}FU + VF\dot{U})] = 0 \quad (20)$$

This equation can also be verified directly using the definitions in eq. (7). Hence from eq.(20), using the cyclic property of the trace, one has

$$i \sum_{SS'} \langle V|U \rangle \text{tr}[G(\dot{V}U + V\dot{U})] = 0 \quad (21)$$

which is the time derivative of the norm (cf. eq.(17)). Next we shall prove that the energy is constant during the time evolution. We need to prove that

$$d \langle \psi | \hat{H} | \psi \rangle / dt = 0 \quad (22)$$

since the norm of the wave function is a constant. Let us set

$$\mathcal{H}[V, U] = \langle \psi | \hat{H} | \psi \rangle, \quad \mathcal{O}[V, U] = \langle \psi | \psi \rangle \quad (23)$$

The Lagrangian associated to EOM1 can be rewritten as

$$\mathcal{L}_1 = i \sum_a \frac{\partial \mathcal{O}}{\partial U_a} \dot{U}_a - \mathcal{H} \quad (24)$$

where  $a = (i\alpha S)$  for brevity. EOM1 can then be written as

$$i \sum_a \frac{\partial^2 \mathcal{O}}{\partial V_b \partial U_a} \dot{U}_a = \frac{\partial \mathcal{H}}{\partial V_b} \quad (25)$$

for all  $b = (\beta j S')$ . Similarly the Lagrangian associated with EOM2 can be recast as

$$\mathcal{L}_2 = -i \sum_b \frac{\partial \mathcal{O}}{\partial V_b} \dot{V}_b - \mathcal{H} \quad (26)$$

and EOM2 can be recast as

$$-i \sum_b \frac{\partial^2 \mathcal{O}}{\partial V_b \partial U_a} \dot{V}_b = \frac{\partial \mathcal{H}}{\partial U_a} \quad (27)$$

Multiplying eq.(25) by  $\dot{V}_b$  and summing over the indices and multiplying eq.(27) by  $\dot{U}_a$  and summing over indices and subtracting the two results we obtain

$$\sum_a \frac{\partial \mathcal{H}}{\partial U_a} \dot{U}_a + \sum_b \frac{\partial \mathcal{H}}{\partial V_b} \dot{V}_b = 0$$

which is precisely the time derivative of  $\mathcal{H}$ . These two constants of motion are a valuable test to check whether the equations of motion have been integrated to reasonable accuracy.

### 3 A brief description of the numerical method.

We solve numerically EOM2 (eq.15) and eq.(16b)). The reason behind this choice is that the subroutines which are needed can be extracted from HMD computer

programs, which have been tested to very high accuracy. As pointed out in the previous section, it is not advisable to store the matrix  $L_2$ . However we can easily evaluate  $L_2 v$  where  $v$  is any vector. Actually we can easily evaluate any power of  $L_2$  applied to  $v$ . The linear system of eq.(16b) can then be solved by projecting eq.(16b) into the subspace generated by the vectors  $v, L_2 v, (L_2)^2 v, \dots$  where  $v$  is an arbitrary trial solution of the linear system (known as Krylov subspace), followed by Gram-Schmidt orthonormalization. Since  $L_2$  is hermitian the projection of  $L_2$  gives a tridiagonal matrix (just as in the standard shell model method) and the linear system can then be efficiently solved. We have implemented the so called direct Lanczos method, the full detail of which (including the algorithm) can be found in ref. [9]. In our computer program the iterations stop when the residual vector  $-iL_2 \dot{V} - R_2$  has a norm less than  $10^{-12}$ . In this work we considered  ${}^6\text{Li}$  with the interaction given by the N3LO NN potential renormalized using the Lee-Suzuki method to 4 major harmonic oscillator shells. Once  $\dot{V}$  has been determined we solve the differential equation in time using a rank-4 Runge-Kutta method. The time step is typically  $10^{-3} \text{MeV}^{-1}$ . In fig. 1 we show the errors in the energies and in the overlaps  $DE = E(t) - E(0)$  and  $DO = \ln(\mathcal{O}(t)/\mathcal{O}(0))$  as a function of time. In fig. 1 we took snapshots every 500 (typically) time steps. Typically we ran the time evolution up to few hundreds  $\text{MeV}^{-1}$ . The number of Lanczos iterations needed for convergence depends on the number of Slater determinants. For 1 Slater determinant (TDHF) we need about 4 Lanczos iterations to reach machine accuracy. This number increases as we increases the number of Slater determinants. For example for 3 Slater determinants we need typically 36 Lanczos iterations and for 5 Slater determinants we need about 50 iterations. In all runs we considered  $\hbar\omega = 12 \text{MeV}$  and we added to the Hamiltonian the standard center of

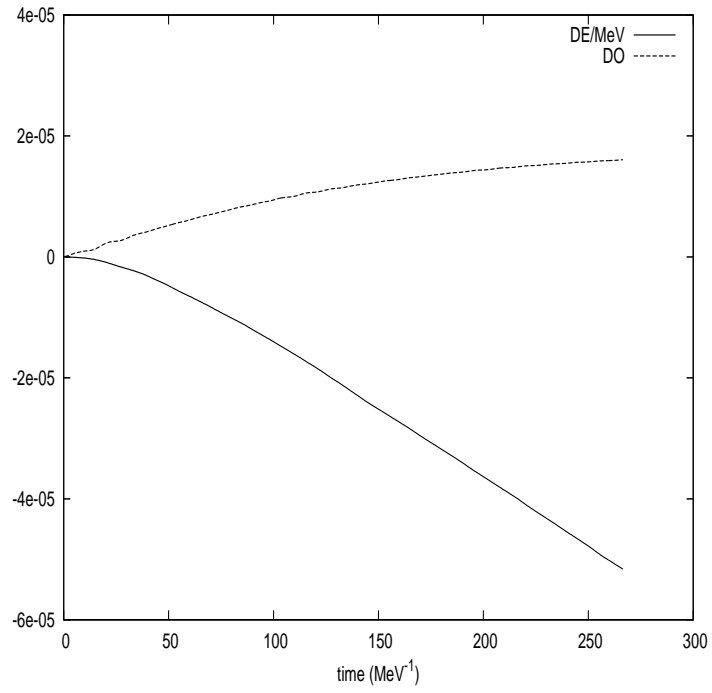


Figure 1: Variation of the energy (in MeV) and variation of the norm as a function of time. We took 3 Slater determinants with 4 major shells at  $\hbar\omega = 12\text{MeV}$  Snapshots are taken every  $0.5\text{MeV}^{-1}$ .

mass Hamiltonian with a strength equal to 1.

## 4 Strength functions.

We evaluate the strength function for a one body operator  $\hat{Q}$

$$S(E) = \sum_n | \langle n | \hat{Q} | 0 \rangle |^2 \delta(E - E_n^*)$$

$E_n^*$  being the excitation energy of the  $n$ -th approximate eigenstate, with the boost method, as follows. First we determine the approximate ground-state of the system  $|0\rangle$ , then at time  $t = 0^+$  we boost the system with the unitary operator

$$|\psi(0^+)\rangle = \exp(i\eta\hat{Q})|0\rangle$$

for sufficiently small  $\eta$ . We then evolve this wave function in time by solving the equations of motion and evaluate

$$Q(t) = \langle \psi(t) | \hat{Q} | \psi(t) \rangle$$

The strength function can be obtained using the Fourier transform of  $Q(t)$  (see for example ref. [12])

$$\overline{Q}(E) = \int_0^T dt e^{i(E+i\Gamma)t} Q(t)$$

for sufficiently large  $T$  such that  $e^{-\Gamma T}$  is negligible via the relation

$$S(E) = \frac{1}{\eta\pi} \text{Im}(\overline{Q}(E))$$

The width  $\Gamma$  is sufficiently small and such that very high frequency oscillations are smoothed out. We take typically  $\Gamma = 0.1 \text{ MeV}$ . We need to evolve the system

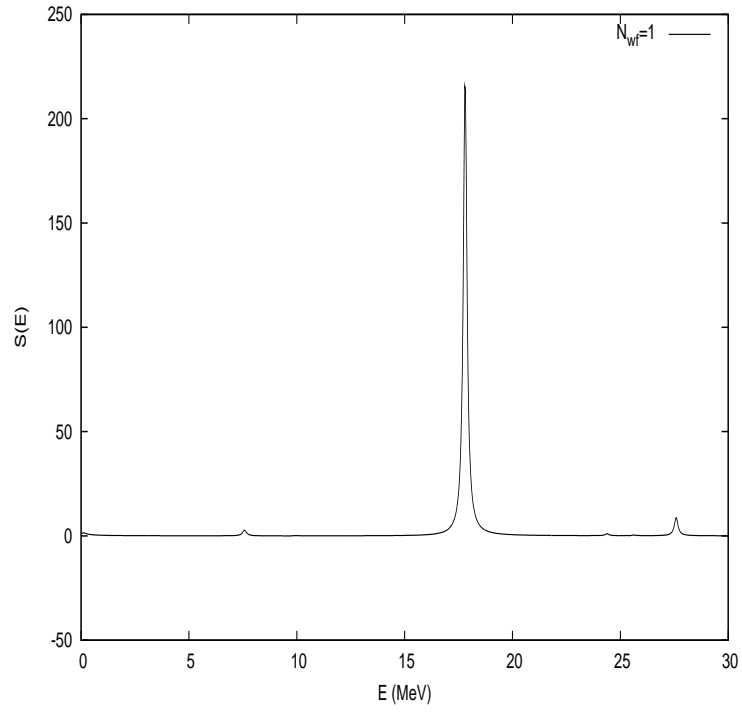


Figure 2: Monopole strength function for  $N_w = 1$  (TDHF).

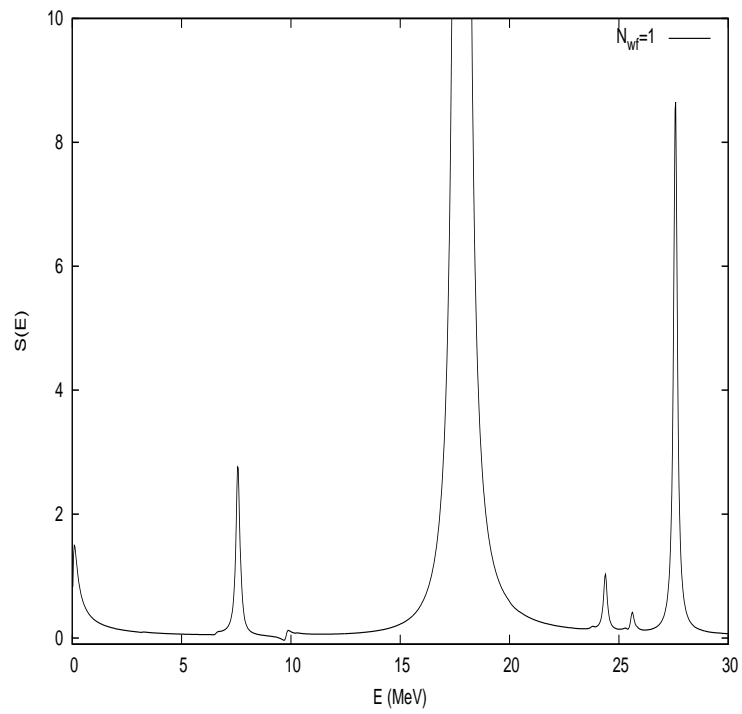


Figure 3: Magnification of fig. 2.

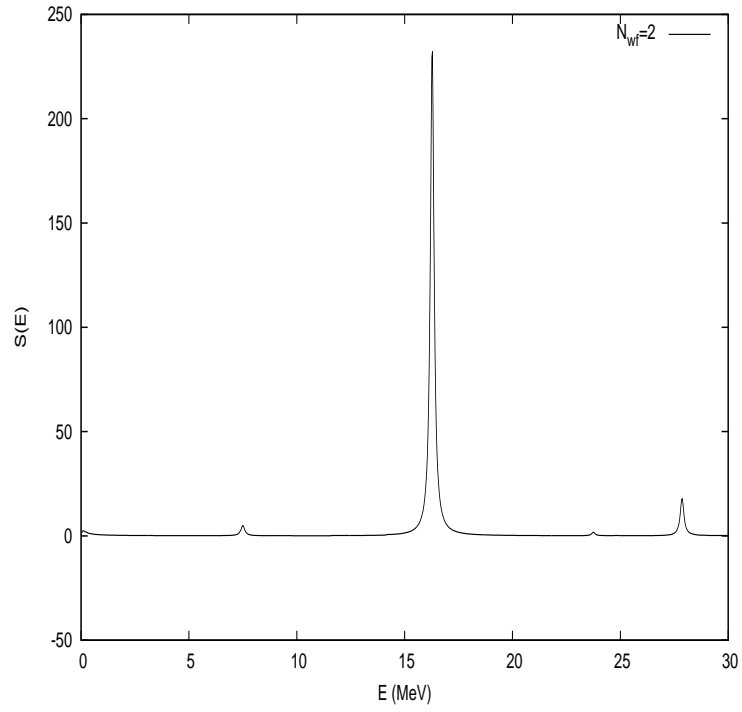


Figure 4: Monopole strength function for  $N_w = 2$  Slater determinants.



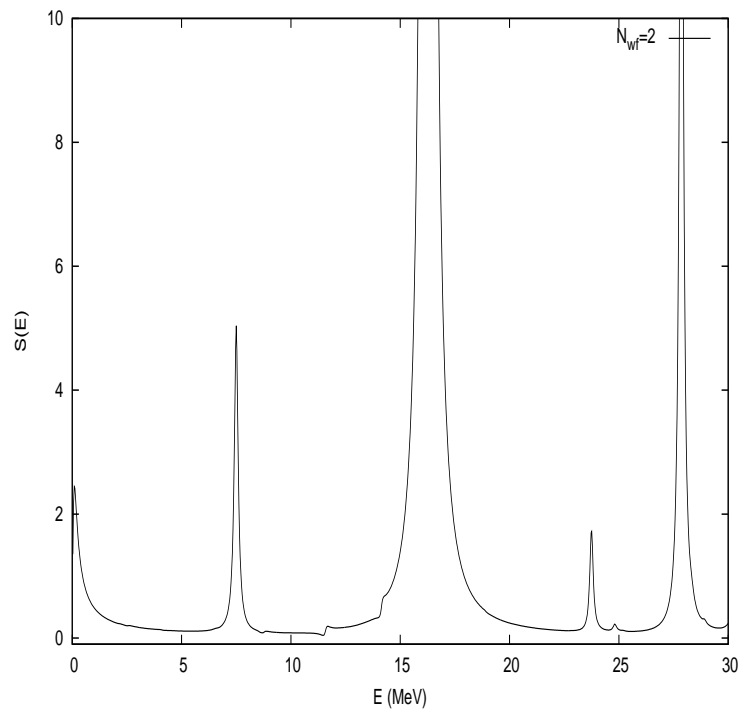


Figure 5: Magnification of fig. 4.

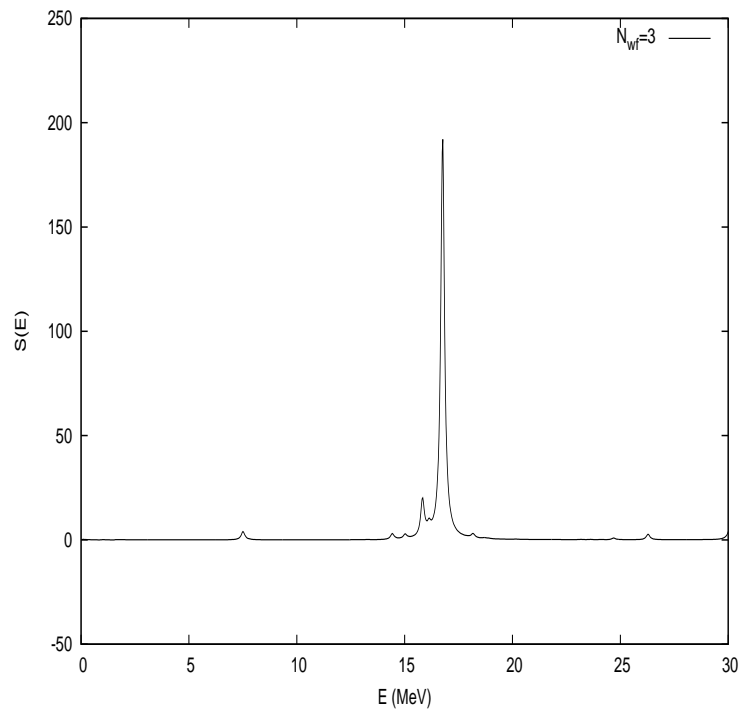


Figure 6: Strength function for  $N_w = 3$  Slater determinants.

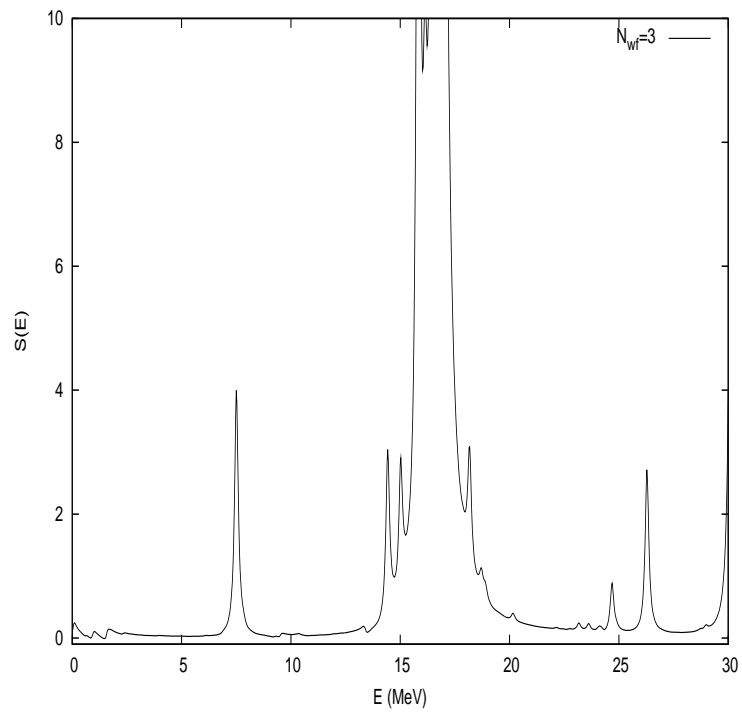


Figure 7: Magnification of fig. 6.

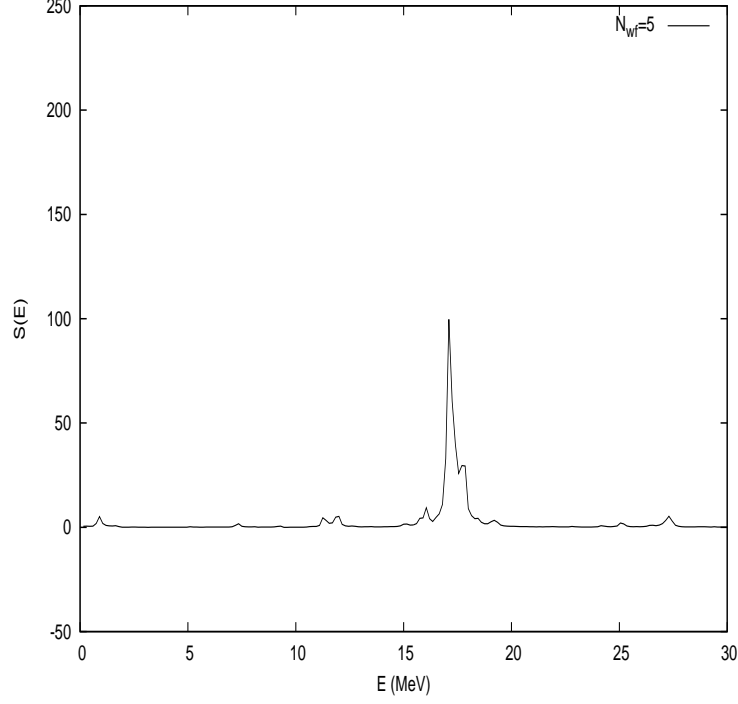


Figure 8: Strength function for  $N_w = 5$  Slater determinants.

after the boost for about  $T = 100 \text{ MeV}^{-1}$ . In this exploratory calculation we have used the isoscalar monopole operator  $Q = r^2$ .

In figs. (2),(3) we show the results obtained in the case 1 Slater determinant (TDHF). Fig (3) is a magnification of fig (2). The strength function is dominated by the dominant peak at  $E \simeq 17.8 \text{ MeV}$ . Some weaker peaks can be seen at  $E \simeq 0.1 \text{ MeV}$ ,  $E \simeq 7.56 \text{ MeV}$  and  $E \simeq 27.6 \text{ MeV}$ . Using 2 Slater determinants we obtained the results shown in figs. (4) and (5). The dominant peak is now at  $E \simeq 16.2 \text{ MeV}$ . The secondary maxima are at  $E \simeq 0.1 \text{ MeV}$ ,  $7.5 \text{ MeV}$ ,  $27.8 \text{ MeV}$ , almost on the same position of the TDHF case. Using 3 Slater determinants, we obtained the results of figs, (6) and (7). The main peak at  $E \simeq 16.7 \text{ MeV}$  shows considerable fragmentation around 15 MeV while the secondary peak at  $7.5 \text{ MeV}$

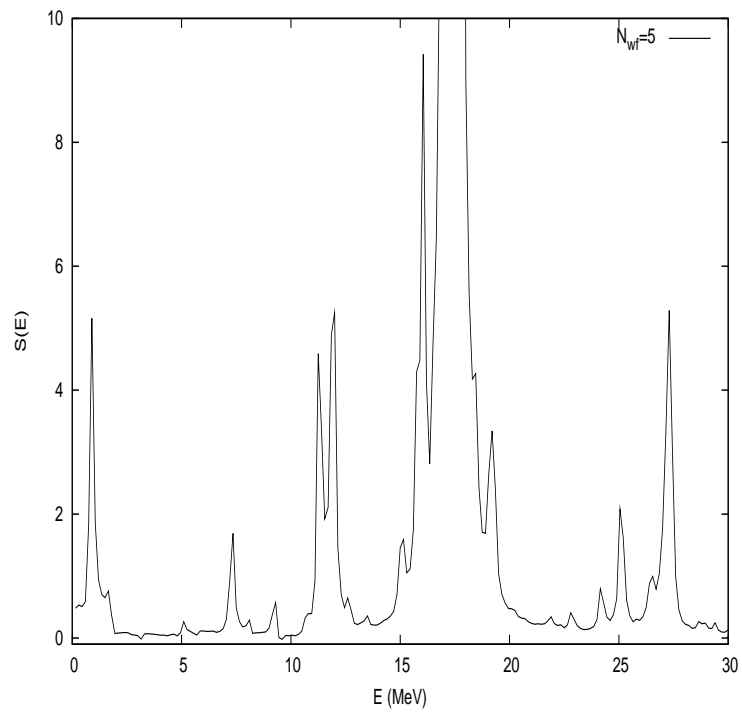


Figure 9: Magnification of fig. 8.

is nearly unchanged. The peak around 27 MeV has nearly disappeared and has moved to lower excitation energies. We considered in this work a maximum of 5 Slater determinants, and the results for the strength function are shown in fig. 8 and the details in fig. 9. The main peak is located at  $17.1\text{MeV}$  closer to the TDHF case. However secondary peaks are at slight different energies and the strength shows a more pronounced fragmentation.

We have not studied the strength function for a larger number of major shells and as a function of the h.o. frequency for increasing number of Slater determinants. Such a study is necessary in order to promote the method as an ab-initio method. Nor we have studied the small amplitude limit of the equations of motion in the case of many Slater determinants which is the equivalent of the RPA with several Slater determinants. Our purpose in this work is to define the method, solve the equations of motion and verify our computer programs. Such studies will be presented in future works.

## References

- [1] P. Ring and P. Schuck. The Nuclear Many-Body Problem. Springer-Verlag New York 1980,
- [2] J.Toivanen, B.G.Carlsson, J.Dobaczewski, K.Mizuyama, R.R.Rodriguez-Guzman,  
P.Toivanen and P.Vesely. Phys. Rev. **C 81**, 034312(2010)
- [3] P.Hoodbhoy and J.W.Negele, Phys. Rev. **C 18**, 2380(1978)

- [4] K.Schoenhammer and O.Gunnarson, Phys. Rev. **B 18** , 6606(1978)
- [5] D.A.Pigg, G.Hagen, H.Nam, and T.Papenbrock. Phys. Rev. **C 86**, 014308(2012).
- [6] G.Puddu. J. Phys. G: Nucl. Part. Phys. **32**,321 (2006).
- [7] G.Puddu. Eur. Phys. J. **A 31**, 163(2007).
- [8] A.K.Kerman, and S.E.Koonin. Annals of Physics 100, 332(1976).
- [9] Y. Saad. Iterative Methods for Sparse Linear Systems.  
Manchester University Press, Manchester 1992.
- [10] K.Suzuki and S.Y.Lee. Progr. Theor. Phys. **64**,2091(1980).
- [11] K.Suzuki. Progr. Theor. Phys. **68**,246(1982).
- [12] M. Tohyama. and A.S. Umar. Phys.Lett. B516, 415(2001).